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Zero-Divisor Graph of Commutative Ring

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Abstract

we recall several results of zero-divisor graphs of commutative rings. We show that $\Gamma(R)$ is always connected with $diam(\Gamma(R)) \leq 3$. Also we determine when $\Gamma(R)$ is a complete graph or a star graph.

Keywords: *commutative ring, zero-divisor, graph, integral domain.*

1. Introduction

Istvan Beck first introduced the concept of relating a commutative ring to a graph. By the definition he gave, every element of the ring R was a vertex in the graph, let R be a commutative ring with non-zero identity. $R^* = R \backslash \{0\}$, The set $Z(R) = \{x \in R | xy = 0 \; for\ some \; 0 \neq y \in R\}$ is the set of zero-divisors of R and $Z^*(R) = Z(R)\{0\}$. $Z(R)$ is finite if and only if either R is finite or an integral domain. We define the zero-divisor graph of R, denoted $\Gamma(R)$, to be simple graph with vertices set being the set of non-zero zero-divisors of R and with (x,y) an edge if and only if $x \neq y$ and $xy = 0$. A subgraph of a graph is any subset of vertices together with any subset of edges containing those vertices. A path of length n from a vertex x to a distinct vertex y is a sequence of $n+1$ distinct vertices $x = v_0, v_1, v_2, \dots, v_n = y$ such that v_i and v_{i+1} are adjacent for $0 \leq i \leq n-1$. Anderson and Livingstons article contains several results that are important for this paper. In Anderson and Livingston [4], the vertex set of $\Gamma(R)$ was chosen to be $Z^*(R)$, set of zero-divisor of R, and the authors studied the interplay between the ring-theoretic properties of a commutative ring R and graph-theoretic properties of $\Gamma(R)$. The zero-divisor graph of a commutative ring has also been studies by several other authors. (see for example [1,2,14,17]). Two vertex x and y of the graph Γ are connected if there is a path in Γ connecting them. If every two vertices are connected, we say that the graph Γ is connected. If x and y are vertices of a graph, we define the distance between x and y, $d(x,y)$, to be the length of a shortest path between them. The diameter of a graph Γ, denoted diam Γ is defined to be the maximum of the distances $d(x,y)$ as x and y vary over all vertices in the graph. A graph is complete if any two distinct vertices are adjacent. In a commutative ring R, the annihilator (ideal) of an element x, denoted $ann(x)$, is the set of those elements y for which $xy = 0$.In terms of the zero-divisor graph, this would be the set of vertices adjacent to x. not that x itself maybe an element of $ann(x)$. A ring is Noetherian if each of its ideals is finitely generated.

1. Main results

In this section, we show that $\Gamma(R)$ is always connected and has small diameter and girth and we determine which complete graphs and star graphs maybe realized as $\Gamma(R)$.We start this section with the following example.

Example 2-1 consider the ring $R = \{0, a, b\}$ with addition and multiplication operation define as following:

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	c	0

And

.	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	b	0	b
c	0	c	0	c

Here $Z^*(R)\{0, a, b\}$ and arcs of $\Gamma(R)$ are ab and cb.

Remark 2-2 Graph $\Gamma(R)$is empty if and only if R be a integral domain.

Theorem 2-3 If R be a commutative ring then $\Gamma(R)$ is finite graph if and only if R be finite or integral domain.

Proof If R be finite, clearly, zero-divisors set R is finite and graph $\Gamma(R)$ is finite.If R be integer domain, then it not has non-zero zero-divisor and it imply that graph be empty.

Let graph $\Gamma(R)$ be finite and non-empty. There exist $0 \neq x, y \in R$, so that $xy = 0$, input $I = Ann(x)$. since $y \in Ann(x)$, there exist I ⊂ Z(R), and for any $r \in R, ry \in R$. now assume R be finite, so there is $i \in I$ such that set $J = \{r \in R | ry = i\}$ is infinite.

Also, $(r - s)y = 0$, for every $r, s \in J$. therefore $Ann(y) \subset Z(R)$ is infinite that it is a contradiction. Hence R is finite.

Theorem 2-4 Given a commutative ring R, the graph $\Gamma(R)$ is connected with diameter at most 3.

Proof Let $x, y \in Z(R)^*$ be distinct. If $xy = 0$, then $d(x, y) = 1$.

Let $xy \neq 0$, if $x^2 = y^2 = 0$, then $x - xy - y$ is a path oflength 2. So $d(x, y) = 2$.If $x^2 = 0$, $y^2 \neq 0$, then there exist $b \in Z(R)^* \backslash \{x, y\}$ such that $by = 0$. If $bx = 0$, then $x - b - y$ is a path of length 2. That $d(x, y) = 2$.If $bx \neq 0$, then $x - bx - y$ is a path of length 2 that $d(x, y) = 2$.

A similar argument holds if $y^2 = 0$and $x^2 \neq 0$.

Now let xy, $x,^2 y^2$be non-zero. then there exist $a, b \in Z(R)^* \backslash \{x, y\}$ such that $ax = by = 0$.If $a = b$, then $x - a - y$ is a path of length2.So let $a \neq b$, if $ab = 0$, then $x - a - b - y$ is a path of length 3, So $d(x, y) \leq 3$.If $ab \neq 0$, then $x - ab - y$ is a path of length 2, So $d(x, y) \leq 3$. Hence $diam(\Gamma(R)) \leq 3$.

Since there are between any vertex one path. So $\Gamma(R)$ is connected.

Theorem 2-5 Let R be a commutative Artirian ring, if $\Gamma(R)$ contains a cycle, Then $\Gamma(R)$ is with diameter at most 4.

Proof Let graph $\Gamma(R)$ contains a cycle.By $[8.theorem\ 8{,}7]$, R is a finite direct product of Artinian local rings.Let R be local with maximal ideal M.

By $[16.theorem\ 8{,}2]$, $M = ann(x)$. For some $x \in M^*$. If there is distinct element $x, y \in M^* - \{x\}$ such that $yz = 0$, then $y - x - z - y$ is a triangle.

Otherwise graph $\Gamma(R)$ is not contain a cycle where it is a contradiction.in this case; $gr\big(\Gamma(R)\big) = 3$.Now, let $R = R_1 \times R_2$, if$|R_1| \geq 3$ and $|R_2| \geq 3$, then choose $a_i \in R_1 - \{0{,}1\}$. Thus $(1{,}0) - (0{,}1) - (a_1, 0) - (0, a_2) - (1{,}0)$ is a square. So in this case;$gr\big(\Gamma(R)\big) \leq 4$. Therefore let $R_1 = Z_2$.

If $|Z(R_2)| \leq 2$, then Γ(R) is not contain a cycle. Where it is a contradiction.So we have $|Z(R_2)| \geq 3$.

Since graph $\Gamma(R)$ is connected. There are distinct element$x, y \in Z(R_2)^*$, such that $xy = 0$.Therefore $(\overline{0}, x) - (\overline{1}, 0) - (\overline{0}, y) - (\overline{0}, x)$ is a triangle so in this case; $gr\big(\Gamma(R)\big) = 3$.Thus in all cases; $gr\big(\Gamma(R)\big) \leq 4$.

Theorem 2-6 If R be a commutative ring, then there is a vertex of $\Gamma(R)$ which is adjacent to every other vertex if and only if $R \cong \mathbb{Z}_2 \times A$, where A is an integral domain or $Z(R)$ is an annihilator ideal and prime.

proof If $R \cong \mathbb{Z}_2 \times A$ where A is a integral domain, then $(x, y)(1{,}0) = 0$, for every $(x, y) \in \mathbb{Z}_2 \times A$.

in the order words, $(1{,}0)$ is a vertex which is adjacent to every vertex. Now, if $Z(R) = Ann(x)$, $x \in R$, then x is adjacent to every other vertex of graph.

Conversely, let a be a vertex where adjacent to every other vertex of graph. We have $Z(R) = Ann(a) \cup \{a\}$.If $a^2 = 0$, then $a \in Ann(a)$, so $Ann(a) = Z(a)$ is a prime ideal of R.If $a^2 \neq 0$, then $a^2 = a$, since otherwise a^2 is a vertex of graph $\Gamma(R)$, where according to theorem supposalis adjacent a.So $aa^2 = 0$.

Now, since $Z(R)\backslash\{a\} \leq Ann(a)$, so $Ann(a)$ is maximal among annihilator ideals and $a^2 \in Ann(a)$, so$Ann(a)$ is prime. Since $a^2 \in Ann(a)$ it imply that $a \in Ann(a)$, so $a^2 =$ 0where it is a contradiction.Therefore $a^2 = a$, thus $R = Ra \oplus R(1-a)$. Now we show that $Ra = \{0, a\}$. Otherwise there is $x \in Ra\{0, a\}$ so that $x = ra (r \in R)$. We have:

$$x(1-a) = ra(1-a) = ra - ra^2 = 0$$

So $x \in Z(R)^*$, but since $ax = (ra)a = ra = x \neq 0$, therefore x is not adjacent to a, where it is a contradiction. So $Ra = \{0, a\}$ and $Ra \cong \mathbb{Z}_2$. On the other hand, $A = R(1-a)$ is a integral domain. Since otherwise, there exist $0 \neq a_1, a_2 \in A$, such that $a_1 a_2 = 0$. Since a is a vertex where adjacent to every other vertex graph. So we have: $(a + a_1)a_2 = 0$. Thus $(a + a_1)$is a vertex of graph where is not adjacent to a where it is a contradiction. Hence $R \cong \mathbb{Z}_2 \times A$, where A is a integral domain.

Corollary 2-7 Let R be finite commutative ring. then there is a vertex of $\Gamma(R)$ which is adjacent to every other vertex if and only if $R \cong \mathbb{Z}_2 \times F$, where F is a finite field, or R is local. Moreover, for some prime P and integer $n \geq 1$, $|\Gamma(R)| = |F| = P^n$ if $R \cong \mathbb{Z}_2 \times F$ and $|\Gamma(R)| = P^n - 1$ if R is local.

Theorem 2-8 Let R be a commutative ring. Then $\Gamma(R)$ is complete if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $xy = 0$ for all $x, y \in Z(R)$.

Proof Let $\Gamma(R)$ be complete graph, there exist $x \in Z(R)$ such that $x^2 \neq 0$, we show that $x^2 = x$.Other wise $x^3 = x^2 \times x = 0$, so $x^2(x + x^2) = 0$ such that $x^2 \neq 0$ so $x + x^2 \in Z(R)$. If $x + x^2 = x$, then $x^2 = 0$ where it is a contradiction. Thus $x + x^2 \neq x$, so $x^2 = x^2 + x^3 = x(x + x^2) = 0$. Since, $\Gamma(R)$ is a complete graph, whereit is a contradiction, again. Hence $x^2 = x$, as in the proof of theorem 2-6, we have $R \cong \mathbb{Z}_2 \times A$, and necessarily $A \cong \mathbb{Z}_2$.

Conversely, If $xy = 0$, for every $x, y \in Z(R)$, so every arbitrary two vertex are adjacent.hence graph is complete. If $\cong \mathbb{Z}_2 \times \mathbb{Z}_2$, graph is complete, clearly.

Corollary 2-9 Let R be a commutative ring such that $diam\left(\Gamma(R)\right) = 1$. Then $R^2 \neq 0$ implies $R \neq Z(R)$.

Proof $R^2 \neq 0$ implies that $R \neq Z(R)$ or that there exist $x, y \in Z(R)$ such that $xy \neq 0$.If the first condition is true, then the claim is proven; if the second is true, then by theorem 2-8,$\cong \mathbb{Z}_2 \times \mathbb{Z}_2$, and hence $R \neq Z(R)$.

Corollary 2-10 Let R be a commutative ring. If $xy = 0$ or $x = y$, then we definition $x{\sim}y$, for any $x, y \in Z(R)$. Relation $\sim$ is an equivalence relation. If and only if $\Gamma(R)$ be a complete graph .

Proof Let $\sim$ be an equivalence relation, so $xy = 0$ or $x = y$ for any arbitrary two vertex $x, y \in Z(R)$. Since arbitrary two vertex are adjacent. So $\Gamma(R)$ is a complete graph.

Conversely, Let $\Gamma(R)$ be complete graph. $x = x$, for every $x \in Z(R)$ so $x{\sim}x$reflex since $\Gamma(R)$ is a complete graph. Now, $xy = 0$, for every $x \in Z(R)$ so $x{\sim}y$ on the other hand, since R is a commutative ring we have: $yx = 0$, for every $x \in Z(R)$ so $y{\sim}x$, symmetric.

Now, for $x, y, z \in Z(R)$,

$x{\sim}y \Rightarrow xy = 0$, $y{\sim}z \Rightarrow yz = 0$.

$xy = 0, yz = 0 \Rightarrow (xy)(yz) = 0$

Since R is a commutative ring. So

$$(yx)(zy) = 0 \Rightarrow y^{-1}(yxzy) = 0 \Rightarrow xzy = 0$$

$$\Rightarrow (xzy)y^{-1} = 0 \Rightarrow xz = 0 \Rightarrow x{\sim}z$$

Therefore it is transitive. Thus $\sim$ is a equivalence relation.

Theorem 2-11 Let R be a finite commutative ring. If $\Gamma(R)$ is complete, then either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or R is local with $char\ R = P$ or $char\ R = P^2$, and $|\Gamma(R)| = p^n - 1$, where p is prime and integer $n \geq 1$.

Proof For a field F, the graph $\Gamma(\mathbb{Z}_2 \times F)$ is not complete unless $F = \mathbb{Z}_2$. Assume that$R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$, according to corollary 2-7, R must be local with maximal ideal M. So $char\ R = P^2$ for prime p and $m \geq 1$.

Now,if $m \geq 3$, then Rhas non-adjacent zero-divisors. That it is a contradiction. Hence $char\ R = P$or $char\ R = P^2$. Since M is a p $-$group $|\Gamma(R)| = p^n - 1$, for some prime p and $n \geq 1$.

Lemma 2-12 Let R be a finite commutative ring. If $\Gamma(R)$ has exactly one vertex adjacent to every other vertex and no other adjacent vertices, then either $R \cong \mathbb{Z}_2 \times F$, where F is a finite field with $|F| \geq 3$ or R is local with maximal ideal Mthat satisfying $R/M \cong \mathbb{Z}_2$, $M^3 = 0$ and $|M^2| \leq 2$.Thus $|\Gamma(R)|$ is either p^n or $2^n - 1$ for some prime p and integer $n \geq 1$.

Proof If $R \cong \mathbb{Z}_2 \times F$, then according corollary 2-7**,** R is local with maximal ideal M. Thus $M = ann(a)$ for a unique $a \in M$.

Let k be the least positive integer with $M^k = 0$. Then $M = ann(b)$ for any nonzero $b \in M^{k-1}$. Hence $M^{k-1} = \{0, a\}$ and $|M^{k-1}/M^k| = 2$, so $R/M \cong \mathbb{Z}_2$ if $k \geq 4$, then $|M^{k-2}| \geq 4$. Hence for any distinct two vertex $x, y \in M^{k-2} - M^{k-1}$, $xy \in M^{2k-4}M^n$.Thus $xy = 0$, that it is a contradiction. Hence $M^3 = 0$ and $|M^2| \leq 2$.

Theorem 2-13 Let R be a finite commutative ring with$|\Gamma(R)| \geq 4$. Then $\Gamma(R)$ is a star graph if and only if $R \cong \mathbb{Z}_2 \times F$ where F is a finite field.

Proof Let $\Gamma(R)$ be a star graph and $R \ncong \mathbb{Z}_2 \times F$. According corollary 2-7 and lemma 2-12, we may assume that (R, M) is local with $|M| = 2^k$, for some $k \geq 3$ and $|M^2| = 2$. Let $M = ann(a)$ and choose distinct $x, y, z, t \in M^* - \{a\}$, then $xy = xz = xt = a$. Since $M^2 = \{0, a\}$, there not are other zero-divisor relations. Hence $x(y - z) = x(y - t) = 0$, since ann(x) has even order and $x \notin ann(x)$ so $ann(x) = \{0, a\}$. Therefore $y - z = y - t = a$ and $z = t$ where it is a contradiction.

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