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The number of fuzzy subgroups of order $p^n q^m$

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Abstract:

In this paper, we determine the number of fuzzy subgroups and the number of maximal chains of $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$ where p, q are distinct primes and n, m are any two natural number.

Keywords: Fuzzy Subgroup; Number Maximal Chain; Distinct

1. Definition:

In this section, an abelian group of order $p^n q^m$**,** where p, q are distinct primes and n, m are any two natural number, is a cyclic group of the form $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$. μ is said to be fuzzy subgroup of group G if $\mu(x+y) \geq \mu(x) \wedge \mu(y)$ and $\mu(x) = \mu(-x)$. Two fuzzy subgroups μ and υ of G are equivalent and denoted by $\mu \sim \upsilon$, if and only if

i. $\forall\, x, y \in G$, $\mu(x) > \mu(y)$ if and only if $\upsilon(x) > \upsilon(y)$,
ii. $\mu(x) = 0$ if and only if $\upsilon(x) = 0$.

Two fuzzy subgroups μ and υ are said to be *distinct* if and only if *,* $\mu \nsim \upsilon$.
A finite n -chain is a collection of numbers on $[0, 1]$ of the form $1 > \lambda_1 > \lambda_2 > \cdots > \lambda_n$where the last entry may not be zero. This is written simply as $1\lambda_1\lambda_2 \cdots \lambda_n$ in the descending order. The numbers $1, \lambda_1, \lambda_2, \cdots, \lambda_n$ are called pins. Further, we say 1 occupies the first position, λ_i occupies the $(i+1)$th position for $1 \leq i \leq n$. Notice that the length of an n -chain is $+1$. thus the number of positions available in an n -chain is equal to the length of the chain which is $n+1$.
An $n+1$-tuple $1\lambda_1\lambda_2 \cdots \lambda_n$ of real numbers in (i) of the form $1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, is called a *keychain*. An increasing maximal chain of $n+1$ subgroups of G starting with trivial subgroup $\{0\}$ is called a *flag* on G.
A *pinned-flag* is a pair (φ, ℓ) where φ is a flag on G and ℓ is a keychain, that for fuzzy subgroups as follows:

$$0^1 \subset G_1^{\lambda_1} \subset G_2^{\lambda_2} \subset \cdots \subset G_n^{\lambda_n}$$

The correspondence between pinned-flags and equivalence classes of fuzzy subgroups is clear that if we let a fuzzy subgroup μ correspondence to (φ, ℓ) on G as follows:

$$\mu(x) = \begin{cases} 1 & x = 0 \\ \lambda_1 & x \in G_1 \backslash \{0\} \\ \lambda_2 & x \in G_2 \backslash G_1 \\ \vdots & \vdots \\ \lambda_n & x \in G_n \backslash G_{n-1} \end{cases}$$

where the component G_n is group G. We denote this simply by $G_n^{\lambda_n} = G^{\lambda_n}$. Conversely, given any fuzzy subgroup we can associate a pinned-flag by considering appropriate $\alpha - cuts$.

2. Maximal chains of G

In this section, we determine the number N of maximal chains of $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$ as a summation through m. We develop a method to represent the maximal chains diagrammatically to suit our inductive steps. For any natural number n and $m = 0$, it is obvious that G has only one maximal chain, $0 \subset \mathbb{Z}_p \subset \mathbb{Z}_{p^2} \subset \cdots \subset \mathbb{Z}_{p^{n-1}} \subset \mathbb{Z}_{p^n}$, which we see in Fig. 1.

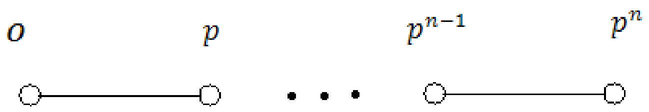

Fig. 1

Proposition 2.1: Let $G = \mathbb{Z}_{p^n} + Z_q$ where p and q are distinct primes. Then the number of maximal chains of G is $n + 1$.

Proof: In Fig. 2 we have drawn the crisp subgroup lattice diagram of G. The result is clear from the diagram and we see that G has $n + 1$ maximal chains.

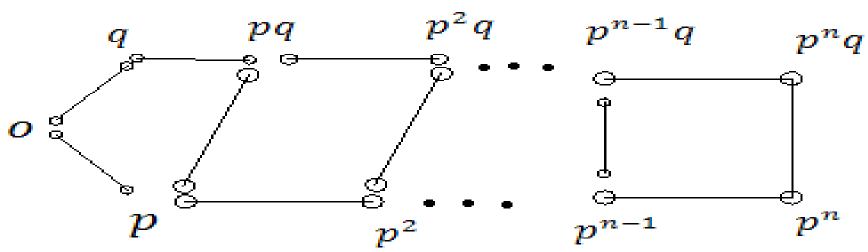

Fig. 2

Proposition 2.2: Let $G = \mathbb{Z}_{p^2} + \mathbb{Z}_{q^m}$ where p and q are distinct primes and $m \geq 2$. Then the number of maximal chains of G is $\sum_{i=-1}^{m-1}(m-i)$.

Proof: Let $m = 2$. We observe that $G = \mathbb{Z}_{p^2} + \mathbb{Z}_{q^2}$ has six maximal chains,

$0 \subset p \subset p^2 \subset p^2q \subset p^2q^2$

$0 \subset p \subset pq \subset p^2q \subset p^2q^2$

$0 \subset p \subset qp \subset pq^2 \subset p^2q^2$

$0 \subset q \subset q^2 \subset pq^2 \subset p^2q^2$

$0 \subset q \subset pq \subset pq^2 \subset p^2q^2$

$0 \subset q \subset pq \subset p^2q \subset p^2q^2$

Also

$$\sum_{i=-1}^{2-1}(2-i) = \sum_{i=-1}^{1}(2-i) = (2+1)+(2)+(2-1) = 6$$

Now suppose the result is true for some positive integer k. Fig. 3 shows how to determine the number of maximal chains of $\mathbb{Z}_{p^2} + \mathbb{Z}_{p^k}$.

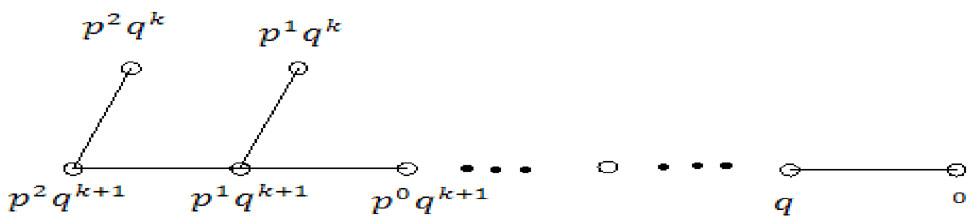

Fig. 3

From the above diagram, p^2q^k has $\sum_{i=-1}^{k-1}(k-i)$ maximal chains and pq^k has $k+1$ maximal chains. q^{k+1} has one maximal chain. Thus $G = \mathbb{Z}_{p^2} + \mathbb{Z}_{q^{k+1}}$ has $\sum_{i=-1}^{k-1}(k-i)$ plus $k+1+1$ maximal chains which can clearly be seen to be equal to

$\sum_{i=-1}^{k-1}(k-i) + k + 1 + 1 = \sum_{i=-1}^{k-1}(k-i) + k + 2 = \sum_{i=-1}^{k}(k+1-i).$

This completes the induction.

Proposition 2.3: Let $G = \mathbb{Z}_{p^3} + \mathbb{Z}_{q^m}$ where p and q are distinct primes. Then the number of maximal chains of G is $\sum_{i=-1}^{m-1} r_i(m-i)$ where $r_i = \frac{(3+i-1)!}{(3-2)!(1+i)!}$, $m \geq 2$.

Proof: Let $m = 2$. We observe that $G = \mathbb{Z}_{p^3} + \mathbb{Z}_{q^2}$ has ten maximal chains,

$$N = \sum_{i=-1}^{1} r_i(2-i) = r_{-1}(2+1) + r_0(2-0) + r_1(2-1)$$
$$= \frac{(3-1-1)!}{(3-2)!\,(1-1)!}(3) + \frac{(3-1)!}{(3-2)!\,1!}(2) + \frac{(3+1-1)!}{(3-2)!\,(1+1)!}(1)$$
$$= 3 + 2 \times 2 + 3 = 10$$

let the result is true for some positive integer k. Fig. 4 shows how to determine the number of maximal chains of $= \mathbb{Z}_{p^3} + \mathbb{Z}_{q^k}$.

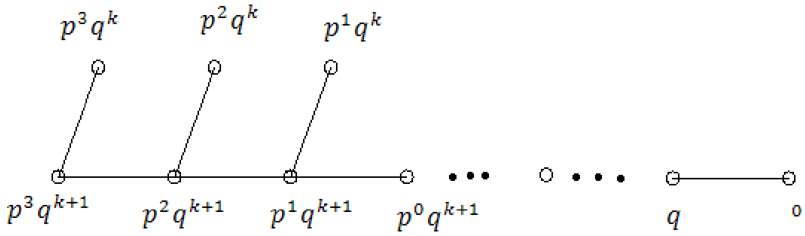

Fig. 4

From the above diagram, p^3q^k has $\sum_{i=-1}^{k-1} r_i(k-i)$ maximal chains and pq^k has $k+1$ maximal chains. q^{k+1} has one maximal chain. Thus G has $\sum_{i=-1}^{k-1} r_i(k-i)$ plus $k+1+1$ maximal chains which can clearly be seen to be equal to

$\sum_{i=-1}^{k} r_i(k+1-i) = \sum_{i=-1}^{k} r_i(k-i) + \sum_{i=-1}^{k} r_i = \sum_{i=-1}^{k-1} r_i(k-i) + r_k(k-k) + \sum_{i=-1}^{k} r_i = \sum_{i=-1}^{k-1} r_i(k-i) + \sum_{i=-1}^{k} \frac{(3+i-1)!}{(3-2)!(1+i)!} = \sum_{i=-1}^{k-1} r_i(k-i) + \sum_{i=-1}^{k} \frac{(i+2)!}{(i+1)!} = \sum_{k=-1}^{k-1} r_i(k-i) + \sum_{i=-1}^{k} i+2 = \sum_{i=-1}^{k-1} r_i(k-i) + (1+2+3+4+\cdots+k+2) = \sum_{k=-1}^{k-1} r_i(k-i) + \sum_{i=-1}^{k}(k+1-i)$

This completes the induction.

Theorem 2.4: Let $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$ where p and q are distinct primes. Then the number of maximal chains of G is $\sum_{i=-1}^{m-1} r_i(m-i)$ where $r_i = \frac{(n+i-1)!}{(n-2)!(1+i)!}$,$n \geq 2$.

Proof: For a fixed n we apply induction on m. Now let that $\mathbb{Z}_{p^n} + \mathbb{Z}_{q^k}$ has $\sum_{i=-1}^{k-1} r_i(k-i)$ maximal chains where $r_i = \frac{(n+i-1)!}{(n-2)!(1+i)!}$. We show that $\mathbb{Z}_{p^n} + \mathbb{Z}_{q^{k+1}}$ has $\sum_{i=-1}^{k} r_i(k+1-i)$ maximal chains where $r_i = \frac{(n+i-1)!}{(n-2)!(1+i)!}$.

As in Propositions 2.2 and 2.3, consider Fig. 5, we observe that total maximal chains as follows:

$p^n q^{k+1} \supset p^n q^k \supset \cdots \qquad (n,k)$

$p^n q^{k+1} \supset p^{n-1} q^{k+1} \supset p^{n-1} q^k \cdots \qquad (n-1,k)$

$p^n q^{k+1} \supset p^{n-1} q^{k+1} \supset p^{n-2} q^{k+1} \supset p^{n-2} q^k \cdots \qquad (n-2,k)$

$p^n q^{k+1} \supset p^{n-1} q^{k+1} \supset p^{n-2} q^{k+1} \supset p^{n-3} q^{k+1} \supset p^{n-3} q^k \supset \cdots \qquad (n-3,k)$

$\vdots$

$p^n q^{k+1} \supset p^{n-1} q^{k+1} \supset \cdots \supset p^{n-(n-1)} q^{k+1} \supset p^1 q^k \supset \cdots \qquad (1,k)$

$p^n q^{k+1} \supset p^{n-1} q^{k+1} \supset \cdots \supset p^0 q^{k+1} \supset q^k \supset \cdots \qquad (0,k)$

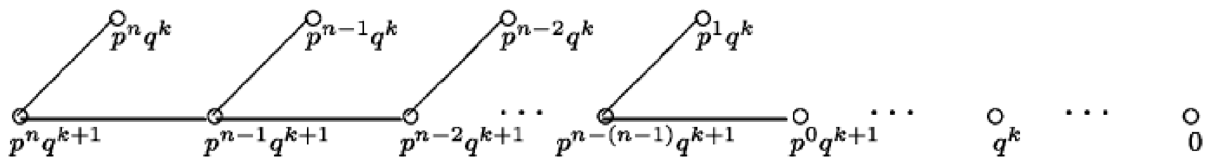

Fig. 5

For $2 \le j \le n$, $(n-j,k)$ yields $\sum_{i=-1}^{k} r_{(n-j)i}(k-i)$ maximal chains that $r_{(n-j)i} = \frac{(n-j+i-1)!}{(n-j-2)!(1+i)!}$. Also $(1,k)$ yields $k+1$ maximal chains and $(0,k)$ has only one maximal chain.

Thus, the total number of maximal chains is the double sum

$$N = \sum_{j=0}^{n-2} \sum_{i=-1}^{k-1} r_{(n-j)i}(k-i) + k + 1 + 1 = \sum_{j=0}^{n-2} \sum_{i=-1}^{k-1} \frac{(n-j+i-1)!}{(n-j-2)!\,(1+i)!}(k-i) + k + 1 + 1$$

$$= \sum_{j=0}^{n-2} \left[\frac{(n-j-2)!}{(n-j-2)!\,0!}(k+1) + \frac{(n-j-1)!}{(n-j-2)!\,1!}k + \frac{(n-j)!}{(n-j-2)!\,2!}(k-1) \right.$$

$$\left. + \frac{(n-j+1)!}{(n-j-2)!\,3!}(k-2) + \cdots + \frac{(n-j+k-2)!}{(n-j-2)!\,k!}.1 \right] + k + 1 + 1 = \sum_{j=0}^{n-2} \left[k+2 \right.$$

$$+ (n-j-1)k + \frac{(n-j)(n-j-1)(k-1)}{2!}$$

$$\left. + \frac{(n-j+1)(n-j)(n-j-1)(k-2)}{3!} + \cdots + \frac{(n-j+k-2)\cdots(n-j-1)}{k!} \right]$$

$$+ k + 1 + 1$$

$$= (n-1)(k+1) + k\sum_{j=0}^{n-2}(n-j-1) + \frac{k-1}{2!}\sum_{j=0}^{n-2}(n-j)\,(n-j-1)$$
$$+ \frac{k-2}{3!}\sum_{j=0}^{n-2}(n-j+1)(n-j)\,(n-j-1) + \cdots$$
$$+ \frac{1}{k!}\sum_{j=0}^{n-2}(n-j+k-2)\cdots(n-j-1) + k + 1 + 1$$

Remark 2.5: For any positive integer n and k,

$$(n)(n+1)\cdots(n+k-1) + (n-1)(n)\cdots(n+k-2) + \cdots + 2.3\cdots(k+1) + 1.2\cdots k$$
$$= \frac{1}{k+1}n(n+1)(n+2)\cdots(n+k)$$

For $k = 2, (n)(n+1) + (n-1)n + \cdots + 1.2 = \frac{1}{3}n(n+1)(n+2)$

For $k = 3, (n)(n+1)(n+2) + (n-1)n(n+1) + \cdots + 1.2.3$

$$= \frac{1}{4}n(n+1)(n+2)(n+3)$$

So, with n replaced by $n-1$

$$\sum_{j=0}^{n-2}(n-j)\,(n-j-1) = (n-1)n + (n-2)(n-1) + \cdots + 1.2 = \frac{1}{3}(n-1)n(n+1)$$

$$\sum_{j=0}^{n-2}(n-j+1)(n-j)\,(n-j-1) = (n-1)n(n+1) + (n-2)(n-1)n + \cdots + 1.2.3$$
$$= \frac{1}{4}(n-1)n(n+1)(n+2)$$

Thus, the total number of maximal chains is

$$N = (n-1)(k+1) + k\frac{n(n-1)}{2} + \frac{k-1}{2!}.\frac{(n-1)n(n+1)}{3} + \frac{k-2}{3!}\frac{(n-1)n(n+1)(n+2)}{4}$$
$$+ \cdots + \frac{1}{k!}.\frac{(n-1)n(n+1)\cdots(n+k-1)}{k+1} + k + 1 + 1$$

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$$\Rightarrow N = (n-1)(k+1) + \frac{k}{2!}n(n-1) + \frac{k-1}{3!}(n-1)n(n+1) + \frac{k-2}{4!}(n-1)n(n+1)(n+2) + \cdots + \frac{1}{(k+1)!}(n-1)n(n+1)\cdots(n+k-1) + k+1+1 \qquad (2.1)$$

On the other hand,

$$\sum_{i=-1}^{k} r_i(k+1-i) = \sum_{i=-1}^{k} \frac{(n+i-1)!}{(n-2)!\,(1+i)!}(k+1-i) = \frac{(n-2)!}{(n-2)!\,0!}(k+2) + \frac{(n-1)!}{(n-2)!\,1!}(k+1) + \frac{n!}{(n-2)!\,2!}k + \cdots + \frac{(n+k-1)!}{(n-2)!\,(1+k)!}.1$$

$$= (k+2) + (n-1)(k+1) + \frac{n(n-1)}{2!}k + \cdots + \frac{(n+k-1)(n+k-2)\cdots(n-1)}{(k+1)!} \qquad (2.2)$$

We see the right-hand side (2.1) and (2.2) are equal. So the number of maximal chains $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$ and $\sum_{i=-1}^{k} r_i(k+1-i)$ where $r_i = \frac{(n+i-1)!}{(n-2)!(1+i)!}$ are equal. This completes the proof.

3. Fuzzy subgroups of G

In this section, given a maximal chain C of subgroups of G, there are $2^{n+1}-1$distinct equivalence classes of fuzzy subgroups. But given two such maximal chains, it is not necessarily true that the number of distinct equivalence classes of fuzzy subgroup is $2(2^{n+1}-1)$. This is because some keychains on distinct maximal chains determine the same equivalence class of fuzzy subgroups. For example, the keychain $1\lambda\lambda\beta\gamma$ on the following two maximal chains

$$0 \subset p \subset pq \subset p^2q \subset p^2q^2 \qquad (3.1)$$

$$0 \subset q \subset pq \subset p^2q \subset p^2q^2 \qquad (3.2)$$

The fuzzy subgroup of maximal chain (3.1) is

$$\mu(x) = \begin{cases} 1 & x = 0 \\ \lambda & x \in p\backslash\{0\} \\ \lambda & x \in pq\ \{p\} \\ \beta & x \in p^2q\backslash\{pq\} \\ \gamma & x \in p^2q\backslash^2\{p^2q\} \end{cases}$$

The fuzzy subgroup of maximal chain (3.2) is

$$\upsilon(x) = \begin{cases} 1 & x = 0 \\ \lambda & x \in q \backslash \{0\} \\ \lambda & x \in pq\{q\} \\ \beta & x \in p^2q\{pq\} \\ \gamma & x \in p^2q^2\{p^2q\} \end{cases}$$

Also

$$\forall\, x \in p^2q^2 \qquad \upsilon(x) = 0 \iff \mu(x) = 0 \iff \gamma = 0$$

And

$$\forall\, x, y \in G \qquad \upsilon(x) > \upsilon(y) \iff \mu(x) > \mu(y)$$

So $\mu \sim \upsilon$ and the same fuzzy subgroups of two maximal chain is pinned-flag

$$0^1 \subset p^\lambda \subset (pq)^\lambda \subset (p^2q)^\beta \subset (p^2q^2)^\gamma.$$

Therefore, the number of distinct fuzzy subgroups of G is fewer than $2^{n+m+1} \sum_{i=-1}^{m-1} r_i(m - i)$ where the number of maximal chains and length of each maximal chain is $n + m + 1$.

Lemma 3.1: The number of fuzzy subgroups on the maximal chains C_{i+1},

$$0 \le i \le n-1, 0 \subset p \subset p^2 \subset \cdots \subset p^i \subset p^iq \subset p^{i+1}q \subset \cdots \subset p^{n-1}q \subset p^nq$$

distinct from any fuzzy subgroup on any other maximal chain is 2^{n+1}.

Also for $i = 0$, the maximal chain C_1 is $0 \subset q \subset pq \subset p^2q \subset \cdots \subset p^{n-1}q \subset p^nq$.

And the maximal chain C_0 is $0 \subset p \subset p^2 \subset \cdots \subset p^n \subset p^nq.$

Proof: consider the fuzzy subgroup on C_{i+1} given by the set φ_{i+1} of keychains of the form $1 \ge \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_i \ge \lambda_{i+1} \ge \lambda_{i+2} \ge \cdots \ge \lambda_{n+1}$. It is clear that there are $\frac{2^{n+2}}{2} = 2^{n+1}$, distinct keychains in φ_{i+1}. Any fuzzy subgroup μ given by an element of φ_{i+1} on C_{i+1} for any fixed $1 \le i_0 \le n-1$ is clearly distinct from any fuzzy subgroup on C_0 because μ is represented by … $\subset \left(p^{i_0}q\right)^{\lambda_{i_0+1}} \subset \cdots$ which cannot figure in any fuzzy subgroup on C_0. μ is distinct from any fuzzy subgroup υ on C_j for, υ would be represented by… $\subset (p^i)^{\lambda_i} \subset \cdots$ which cannot appear in μ . This completes the proof.

Lemma 3.2: The number of fuzzy subgroups on the maximal chain C_{ij},

$$0 \subset p \subset \cdots \subset p^i \subset p^iq \subset p^{i+1}q \subset \cdots \subset p^jq \subset p^jq^2 \subset \cdots \subset p^{n-1}q^2 \subset p^nq^2$$

distinct from any fuzzy subgroup on any other maximal chain is 2^{n+1}, for $0 \leq i,j \leq n-1$.

Also, for $i = 0$, the maximal chain C_{0j} is

$$0 \subset q \subset pq \subset \cdots \subset p^j q \subset p^j q^2 \subset \cdots \subset p^n q^2$$

And for $j = n-1$ the maximal chain $C_{i(n-1)}$ is

$$0 \subset q \subset q^2 \subset \cdots \subset p^i q^2 \subset \cdots \subset p^{n-1} q^2 \subset p^n q^2$$

Proof: Consider the fuzzy subgroups on C_{ij} given by the set φ_{ij} of keychains of the from $1 \geq \lambda_1 \geq \cdots \geq \lambda_i > \lambda_{i+1} \geq \lambda_{i+2} \geq \cdots \geq \lambda_j > \lambda_{j+1} \geq \cdots \geq \lambda_{n+2}$.

There are $\frac{2^{n+3}}{4} = 2^{n+1}$ distinct keychains in φ_{ij}. Any fuzzy subgroup μ given by an element of φ_{ij} on C_{ij} for any fixed $0 \leq i_0, j_0 \leq n-1$ is distinct from any fuzzy subgroup on C_0 , $0 \subset p \subset p^2 \subset \cdots \subset p^n \subset p^n q$

Because μ is represented by … $\subset (p^{i_0} q)^{\lambda_{i_0+1}} \subset \cdots$ which cannot figure in any fuzzy subgroup on C_0. μ is distinct from any fuzzy subgroup on C_k

$$0 \subset p \subset p^2 \subset \cdots \subset p^k \subset p^k q \subset p^{k+1} q \subset \cdots \subset p^{n-1} q \subset p^r q^s \subset p^n q^2.$$

Where either $r = n, s = 1$ or $r = n-1, s = 2$ and for $i_0 + 1 < k \leq n-1$. if $1 \leq k \leq i_0 + 1$, then μ is distinct from any fuzzy subgroup ν on C_k for, υ would be represented by … $\subset (p^{k-1} q)^{\lambda_k} \subset \cdots$ which cannot figure in μ .

μ is distinct from any fuzzy subgroup on C_{kl} for any $i_0 < k < l \leq n-1$, because μ is represented by … $\subset (p^{i_0} q)^{\lambda_{i_0+1}} \subset \cdots$ which cannot figure in any fuzzy subgroup on C_{kl}. Also, if $0 \leq k, l \leq i_0$ then μ is distinct from any fuzzy subgroup υ on C_{kl} because υ has in its representation … $\subset (p^{k-1} q)^{\lambda_k} \subset \cdots$ which cannot figure in . This completes the proof.

Proposition 3.4: The number of distinct fuzzy subgroup of $G = \mathbb{Z}_{p^n} + \mathbb{Z}_q$is

$2^{n+1+1} \sum_{r=0}^{1} 2^{-r} \binom{n}{n-r} \binom{1}{r} - 1$, where $n \geq 1$.

Proof: By proposition 2.1, the number of maximal chain of G is equal to $n+1$. Only one maximal chain C_0 goes through p^n and is given by

$$0 \subset p \subset p^2 \subset \cdots \subset p^n \subset p^n q.$$

The length of this maximal chain is $n+2$ and therefore the number of fuzzy subgroups on C_0 is

$2^{n+2}-1.$ (3.3)

All other n maximal chains $C_n, , \dots C_2, C_1$ go through $p^{n-1}q$. Each one of these maximal chains can be distinguished from each other by writing $p^{n-1}q$ as
we mean the maximal chain $C_{i+1}, 0 \subset p^{n-1}q$, … , p^2qp^{n-3}, p^1qp^{n-2}, qp^{n-1}. By p^iqp^{n-i-1}
$p \subset p^2 \subset \cdots \subset p^i \subset p^iq \subset p^{i+1}q \subset \cdots \subset p^{n-1}q.$

By lemma 3.1, the number of fuzzy subgroups on C_{i+1} is 2^{n+1}. So, the total number of fuzzy subgroups on these n distinct maximal chains is

$n2^{n+1}.$ (3.4)

On the other hand,

$$2^{n+1+1}\sum_{r=0}^{1} 2^{-r}\binom{n}{n-r}\binom{1}{r}-1 = 2^{n+2}\left(2^0\binom{n}{n}\binom{1}{0}+2^{-1}\binom{n}{n-1}\binom{1}{1}\right)-1$$

$$=2^{n+2}\left(1+\frac{1}{2}n\right)-1=n2^{n+1}+2^{n+2}-1 \quad (3.5)$$

therefore formula (3.5) with the sum of numbers found above in (3.3) and (3.4) is equal.

This completes the proof.

Proposition 3.5: The number of distinct fuzzy subgroups of $G=\mathbb{Z}_{p^n}+\mathbb{Z}_{q^2}$ is
$2^{n+2+1}\sum_{r=0}^{2} 2^{-r}\binom{n}{n-r}\binom{2}{r}-1$, where $n \geq 2$

Proof: For $m=2$, by theorem 2.4, the number of maximal chains of G is

$$\sum_{i=-1}^{1}\frac{(n+i-1)!}{(n-2)!\,(1+i)!}(2-i)=\frac{(n-2)!}{(n-2)!\,0!}3+\frac{(n-1)!}{(n-2)!\,1!}2+\frac{n!}{(n-2)!\,2!}$$
$$=3+2(n-1)+\frac{n(n-1)}{2}=1+2n+\frac{n(n-1)}{2}$$

One maximal chain C_0 goes through p^n and

$$0 \subset p \subset p^2 \subset \cdots \subset p^n \subset p^nq \subset p^nq^2$$

The length of this maximal chain is $n+3$ and the number of fuzzy subgroups on C_0, by proposition 2.3, on [9], is

$2^{n+3}-1$ (3.6)

For $0 \leq i \leq n-1$, maximal chain in the following:

$$0 \subset p \subset p^2 \subset \cdots \subset p^i \subset p^i q \subset p^{i+1} q \subset \cdots \subset p^{n-1} q \subset p^n q \subset p^n q^2$$

and

$$0 \subset p \subset p^2 \subset \cdots \subset p^i \subset p^i q \subset p^{i+1} q \subset \cdots \subset p^{n-1} q \subset p^{n-1} q^2 \subset p^n q^2$$

By lemma 3.1, the number of fuzzy subgroups on anyone of the above maximal chain is 2^{n+2}. So the total number of fuzzy subgroups on these $2n$ distinct maximal chains is

$$2n2^{n+2} \qquad (3.7)$$

There are $\frac{n(n-1)}{2}$ maximal chains passing through $p^{n-1}q^2$, that maximal chain in the following:

$$0 \subset p \subset p^2 \subset \cdots \subset p^i \subset p^i q \subset p^{i+1} q \subset \cdots \subset p^j q \subset p^j q^2 \subset p^{j+1} q^2 \subset \cdots \subset p^{n-1} q^2 \subset p^n q^2.$$

By lemma 3.2, the number of fuzzy subgroups on anyone of these maximal chain is 2^{n+1}. So the total number of fuzzy subgroups on these $\frac{n(n-1)}{2}$ distinct maximal chains is

$$\frac{n(n-1)}{2} 2^{n+1} \qquad (3.8)$$

Also

$$2^{n+2+1} \sum_{r=0}^{2} 2^{-r} \binom{n}{n-r} \binom{2}{r} - 1 = 2^{n+3} \left(\binom{n}{n} \binom{2}{0} + \frac{1}{2} \binom{n}{n-1} \binom{2}{1} + \frac{1}{4} \binom{n}{n-2} \binom{2}{2} \right) - 1$$

$$= 2^{n+3} \left(\frac{n!}{n!\,0!} \times \frac{2!}{2!\,0!} + \frac{1}{2} \times \frac{n!}{(n-1)!\,1!} \times \frac{2!}{1!\,1!} + \frac{1}{4} \times \frac{n!}{(n-2)!\,2!} \times \frac{2!}{2!\,0!} \right) - 1$$

$$= 2^{n+3} \left(1 + n + \frac{n(n-1)}{8} \right) - 1$$

$$= 2^{n+3} - 1 \quad + n2^{n+3} \quad + 2^{n+1} \frac{n(n-1)}{2} \qquad (3.9)$$

Therefore formula (3.9) is equal with the sum of numbers found above in (3.6), (3.7) and (3.8).

This completes the proof.

Theorem 3.6: The number of distinct fuzzy subgroups of $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$ is $2^{n+m+1} \sum_{r=0}^{m} 2^{-r} \binom{n}{n-r} \binom{m}{r} - 1$, where $m \leq n$

Proof : for m=0, by [4], the number of distinct fuzzy subgroups of order p^n in G is equal $2^{n+1} - 1$.

On the other hand,

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$$2^{n+0+1}\sum_{r=0}^{0}2^{-r}\binom{n}{n-r}\binom{0}{r}-1=2^{n+1}\binom{n}{n-0}\binom{0}{0}-1$$
$$=2^{n+1}\times\frac{n!}{n!0!}\times\frac{0!}{0!0!}-1=2^{n+1}-1$$

By theorem 2.4 for any positive integer m, gives the number of maximal chains of $G=\mathbb{Z}_{p^n}+\mathbb{Z}_{q^m}$

For any $r=k, 0\le k\le m$, the $2^{n+k+1}\sum_{r=0}^{k}2^{-r}\binom{n}{n-r}\binom{k}{r}-1.$

We represents the total number of maximal chains passing through $p^{n-k}q^k$ and the length of any maximal chain is $n+m+1$.

So the number of fuzzy subgroups on any one of the maximal chain is $2^{n+m+1}-1$

By to the one used in the above lemmas, Many fuzzy subgroups distinct from any other fuzzy subgroup on any other maximal chain is $\frac{2^{n+m+1}-1+1}{2^k}$

For $0\le r\le k-1$, the number of fuzzy subgroups on other maximal chain passing through $p^{n-r}q^r$.

As r varies over 0 to m, we get the total number of distinct fuzzy subgroups on G

$$2^{n+m+1}\sum_{r=0}^{m}2^{-r}\binom{n}{n-r}\binom{m}{r}-1$$

$$=2^{n+m+1}\left(\binom{n}{n}\binom{m}{0}+\frac{1}{2}\binom{n}{n-1}\binom{m}{1}+\cdots+\frac{1}{2^{m-1}}\binom{n}{n-m+1}\binom{m}{m-1}\right.$$
$$\left.+\frac{1}{2^m}\binom{n}{n-m}\binom{m}{m}\right)-1$$
$$=2^{n+m+1}\left(\frac{n!}{n!\,0!}\times\frac{m!}{m!\,0!}+\frac{1}{2}\times\frac{n!}{(n-1)!\,1!}\times\frac{m!}{(m-1)!\,1!}+\cdots+\frac{1}{2^{m-1}}\right.$$
$$\left.\times\frac{n!}{(m-1)!\,(n-m+1)!}\times\frac{m!}{(m-1)!\,1!}+\frac{1}{2^m}\times\frac{n!}{(n-m)!\,m!}\times\frac{m!}{m!\,0!}\right)-1=\cdots$$
$$=\frac{2^{n+m+1}}{2^k}$$

This completes the proof.

Example 3.7: let $G=\mathbb{Z}_p$, then $0\subset\mathbb{Z}_p$ has the maximal chain of length 2. The keychain 1 1 represents the crisp trivial subgroup of $\mathbb{Z}_p$ and the fuzzy subgroup as follows:

$$\mu(x) = \begin{cases} 1 & x = 0 \\ 1 & x \neq 0 \end{cases}$$

The keychain 1 0 represents the zero trivial subgroup of $\mathbb{Z}_p$ and the fuzzy subgroup as follows:

$$\upsilon(x) = \begin{cases} 0 & x = 0 \\ 0 & x \neq 0 \end{cases}$$

And the 1 λ keychain represents fuzzy subgroup as follows:

$$\beta(x) = \begin{cases} 1 & x = 0 \\ \lambda & x \neq 0 \end{cases}$$

It is easily to see that these are the only distinct equivalence classes of fuzzy subgroups on $\mathbb{Z}_p$. Since the only crisp trivial subgroups of $\mathbb{Z}_p$are 0 and $\mathbb{Z}_p$itself. Therefore $\mathbb{Z}_p$has three the distinct fuzzy subgroups and this is equal

$$2^{n+1} - 1 = 2^{1+1} - 1 = 4 - 1 = 3.$$

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