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Maximal chains of fuzzy groups

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Abstract:

In this paper, we determine the number of maximal chains of $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$ *where* p, q *are distinct primes and* n, m *are any two natural number. Also we propose a combinatorial formula for the number of chain maximal of* G.

Keywords: *Fuzzy Subgroup; Number Maximal Chain; Distinct prime*

1. Introduction:

In this paper, an abelian group of order $p^n q^m$, where p, q are distinct primes and n, m are any two natural number, is a cyclic group of the form $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$. This is a well-known fact in group theory [3]. Since the notion of fuzzy sets was introduction by Zadeh in 1965 [16], there have been attempts to extend useful mathematical notions to this wider setting replacing sets by fuzzy sets [4] and [7]. Mordeson [11] determined a complete system of invariants which are unique to finitely generated, then the classification becomes relatively simpler. In another paper dealing with the number of distinct fuzzy subgroups of a finite abelian group G [13], see also [1], we determined the number of fuzzy subgroups of $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$ where p, q are distinct primes and n, m are any two natural number. In paper [14], we investigated the number of fuzzy subgroups of $G = \mathbb{Z}_{p^1} + \mathbb{Z}_{p^2} + \cdots + \mathbb{Z}_{p^m}$ for distinct primes p^i for $i = 1, 2, \ldots, m$.

2. Preliminaries:

In this section, we define an equivalence relation on the set all fuzzy subsets of G based the heuristic principle that two fuzzy sets which maintain the same relativistic membership values of element are essentially equal.

Let G be a group and $a \in G$. The set $< a > = \{a \in G | n \in \mathbb{Z}\}$ is called *cyclic* subgroup generated by a. The group G is called *cyclic group* if there exist an element $a \in G$ such that $G =< a >$. in this case a is called a generator of G. A group G is abelian if $ab = ba$ for all $a, b \in G$. Cyclic groups are abelian, but the converse is not true. We say that G is a p-group then the every element of G is a power order of p, where p is a prime number. A finite group G is a p-group if and only if the order of G is a power of p.

A partially ordered set is a set P with a binary relation $R \subseteq P \times P$ satisfying all of the following conditions.

i. (reflexivity) $(x, x) \in R$ for all $x \in P$
ii. (anti symmetry) $(x, y) \in R$ and $(y, x) \in R \Rightarrow x = y$
iii. (transitivity) $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$

If P is partially ordered by $\leq_P$, $C \subseteq P$ is a chain in P if the restriction of $\leq_P$ to C is linear. If P is a well-founded partial order then every chain in P is a well- order and the supremum of all ordinals which are order types of chain in P. A chain is maximal if no other chain strictly contains it.

Let X be a nonempty set. A fuzzy subset of X is a function μ from X into [0,1]. μ is said to be fuzzy subgroup of group G if $\mu(x + y) \geq \mu(x) \wedge \mu(y)$ and $\mu(x) = \mu(-x)$. Two fuzzy subgroups μ and υ of G are equivalent and denoted by $\mu \sim \upsilon$, if and only if

i. $\forall\, x, y \in G$, $\mu(x) > \mu(y)$ if and only if $\upsilon(x) > \upsilon(y)$,
ii. $\mu(x) = 0$ if and only if $\upsilon(x) = 0$.

Two fuzzy subgroups μ and υ are said to be *distinct* if and only if , $\mu \nsim \upsilon$.
$\mu \sim \upsilon$, if and only if μ and υ have the same set of level subgroups, that is they determine the same chain of subgroups of type. A finite n -chain is a collection of numbers on $[0, 1]$ of the form $1 > \lambda_1 > \lambda_2 > \cdots > \lambda_n$where the last entry may not be zero. This is written simply as $1\lambda_1\lambda_2 \cdots \lambda_n$ in the descending order. The numbers $1, \lambda_1, \lambda_2, \cdots, \lambda_n$ are called pins. Further, we say 1 occupies the first position, λ_i occupies the $(i + 1)$th position for $1 \leq i \leq n$. Notice that the length of an n -chain is $+1$. thus the number of positions available in an n -chain is equal to the length of the chain which is $n + 1$.
An $n + 1$-tuple $1\lambda_1\lambda_2 \cdots \lambda_n$ of real numbers in (i) of the form $1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, is called a *keychain*. An increasing maximal chain of $n + 1$ subgroups of G starting with trivial subgroup $\{0\}$ is called a *flag* on G.
A *pinned-flag* is a pair (φ, ℓ) where φ is a flag on G and ℓ is a keychain, that for fuzzy subgroups as follows:

$$0^1 \subset G_1^{\lambda_1} \subset G_2^{\lambda_2} \subset \cdots \subset G_n^{\lambda_n}$$

The correspondence between pinned-flags and equivalence classes of fuzzy subgroups is clear that if we let a fuzzy subgroup μ correspondence to (φ, ℓ) on G as follows:

$$\mu(x) = \begin{cases} 1 & x = 0 \\ \lambda_1 & x \in G_1 \backslash \{0\} \\ \lambda_2 & x \in G_2 \backslash G_1 \\ \vdots & \vdots \\ \lambda_n & x \in G_n \backslash G_{n-1} \end{cases}$$

where the component G_n is group G. We denote this simply by $G_n^{\lambda_n} = G^{\lambda_n}$. Conversely, given any fuzzy subgroup we can associate a pinned-flag by considering appropriate $\alpha - cuts$.

3. Maximal chains of G

In this section, we determine the number N of maximal chains of $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$ as a summation through m. We develop a method to represent the maximal chains diagrammatically to suit our inductive steps. For any natural number n and $m = 0$, it is obvious that G has only one maximal chain, $0 \subset \mathbb{Z}_p \subset \mathbb{Z}_{p^2} \subset \cdots \subset \mathbb{Z}_{p^{n-1}} \subset \mathbb{Z}_{p^n}$, which we see in Fig. 1.

Fig. 1

Proposition 2.1: Let $G = \mathbb{Z}_{p^n} + \mathbb{Z}_q$ where p and q are distinct primes. Then the number of maximal chains of G is $n + 1$.

Proof: In Fig. 2 we have drawn the crisp subgroup lattice diagram of G. The result is clear from the diagram and we see that G has $n + 1$ maximal chains.

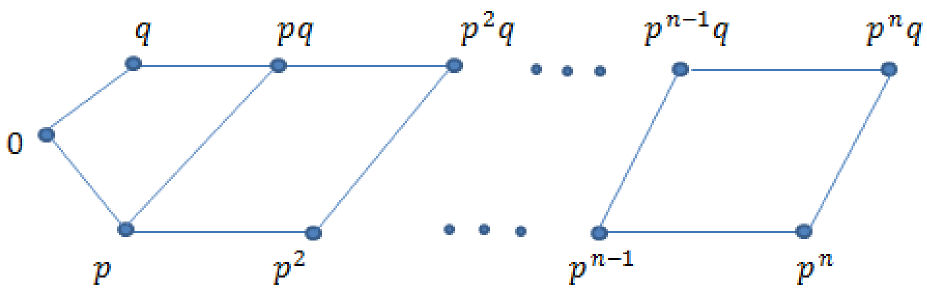

Fig. 2

Example 2.2: Let $G = \mathbb{Z}_{2^2} + \mathbb{Z}_3$. Then the number of maximal chains of G is three.

$0 \subseteq 2 \subseteq 2^2 \subseteq 2^2 \times 3$

$0 \subseteq 2 \subseteq 2 \times 3 \subseteq 2^2 \times 3$

$0 \subseteq 3 \subseteq 3 \times 2 \subseteq 2^2 \times 3$

Also

$$\sum_{i=-1}^{1-1}(1-i) = \sum_{i=-1}^{2}(1-i) = (1+1) + (1) = 3$$

Fig. 3 shows how to determine the number of maximal chains of $G = \mathbb{Z}_{2^2} + \mathbb{Z}_3$

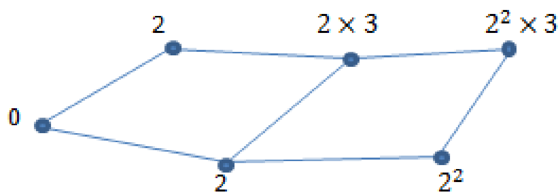

Fig. 3

Proposition 2.3: Let $G = \mathbb{Z}_{p^2} + \mathbb{Z}_{q^m}$ where p and q are distinct primes and $m \geq 2$. Then the number of maximal chains of G is$\sum_{i=-1}^{m-1}(m-i)$.

Proof: Let $m = 2$. We observe that $G = \mathbb{Z}_{p^2} + \mathbb{Z}_{q^2}$has six maximal chains,

$0 \subset p \subset p^2 \subset p^2 q \subset p^2 q^2$

$0 \subset p \subset pq \subset p^2 q \subset p^2 q^2$

$0 \subset p \subset qp \subset pq^2 \subset p^2q^2$

$0 \subset q \subset q^2 \subset pq^2 \subset p^2q^2$

$0 \subset q \subset pq \subset pq^2 \subset p^2q^2$

$0 \subset q \subset pq \subset p^2q \subset p^2q^2$

Also

$$\sum_{i=-1}^{2-1}(2-i) = \sum_{i=-1}^{1}(2-i) = (2+1)+(2)+(2-1) = 6$$

Now suppose the result is true for some positive integer k. Fig. 43 shows how to determine the number of maximal chains of $\mathbb{Z}_{p^2} + \mathbb{Z}_{p^k}$.

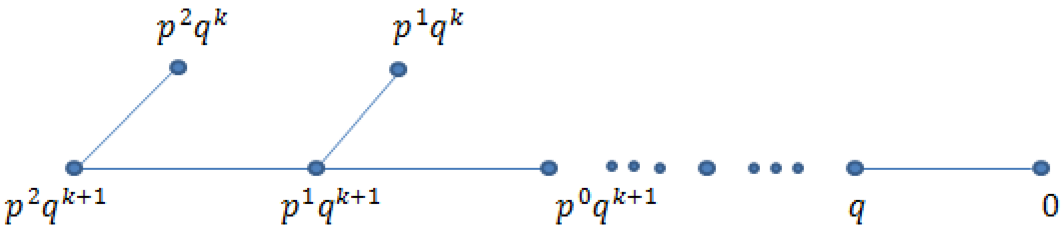

Fig. 4

From the above diagram, p^2q^k has $\sum_{i=-1}^{k-1}(k-i)$ maximal chains and pq^k has $k+1$ maximal chains. q^{k+1}has one maximal chain. Thus $G = \mathbb{Z}_{p^2} + \mathbb{Z}_{q^{k+1}}$ has $\sum_{i=-1}^{k-1}(k-i)$ plus $k+1+1$ maximal chains which can clearly be seen to be equal to

$\sum_{i=-1}^{k-1}(k-i) + k + 1 + 1 = \sum_{i=-1}^{k-1}(k-i) + k + 2 = \sum_{i=-1}^{k}(k+1-i)$.

This completes the induction.

Proposition 2.4: Let $G = \mathbb{Z}_{p^3} + \mathbb{Z}_{q^m}$ where p and q are distinct primes. Then the number of maximal chains of G is $\sum_{i=-1}^{m-1} r_i(m-i)$ where $r_i = \frac{(3+i-1)!}{(3-2)!(1+i)!}$, $m \geq 2$.

Proof: Let $m = 2$. We observe that $G = \mathbb{Z}_{p^3} + \mathbb{Z}_{q^2}$ has ten maximal chains,

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$$N = \sum_{i=-1}^{1} r_i(2-i) = r_{-1}(2+1) + r_0(2-0) + r_1(2-1)$$
$$= \frac{(3-1-1)!}{(3-2)!\,(1-1)!}(3) + \frac{(3-1)!}{(3-2)!\,1!}(2) + \frac{(3+1-1)!}{(3-2)!\,(1+1)!}(1)$$
$$= 3 + 2\times 2 + 3 = 10$$

let the result is true for some positive integer k. Fig. 5 shows how to determine the number of maximal chains of $= \mathbb{Z}_{p^3} + \mathbb{Z}_{q^k}$.

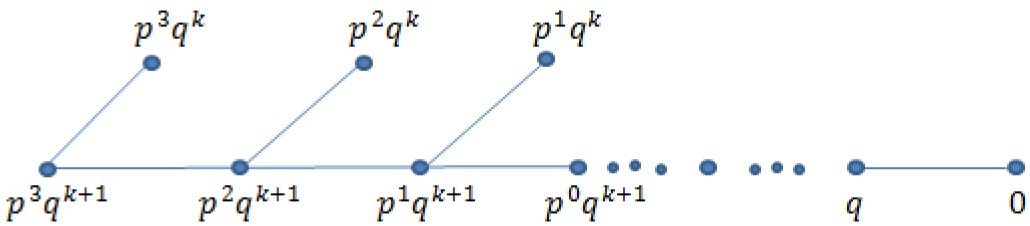

Fig. 5

From the above diagram, p^3q^k has $\sum_{i=-1}^{k-1} r_i(k-i)$ maximal chains and pq^k has $k+1$ maximal chains. q^{k+1} has one maximal chain. Thus G has $\sum_{i=-1}^{k-1} r_i(k-i)$ plus $k+1+1$ maximal chains which can clearly be seen to be equal to

$\sum_{i=-1}^{k} r_i(k+1-i) = \sum_{i=-1}^{k} r_i(k-i) + \sum_{i=-1}^{k} r_i = \sum_{i=-1}^{k-1} r_i(k-i) + r_k(k-k) + \sum_{i=-1}^{k} r_i = \sum_{i=-1}^{k-1} r_i(k-i) + \sum_{i=-1}^{k} \frac{(3+i-1)!}{(3-2)!(1+i)!} = \sum_{i=-1}^{k-1} r_i(k-i) + \sum_{i=-1}^{k} \frac{(i+2)!}{(i+1)!} = \sum_{k=-1}^{k-1} r_i(k-i) + \sum_{i=-1}^{k} i+2 = \sum_{i=-1}^{k-1} r_i(k-i) + (1+2+3+4+\cdots+k+2) = \sum_{k=-1}^{k-1} r_i(k-i) + \sum_{i=-1}^{k}(k+1-i)$

This completes the induction.

Example 2.5: Let $G = \mathbb{Z}_{2^3} + \mathbb{Z}_{3^2}$ where $p = 2$ and $q = 3$ are distinct primes and $n = 3\,, m = 2$. Then the number of maximal chains of G is ten.

The group $\mathbb{Z}_{2^3} + \mathbb{Z}_{3^2}$ has the following maximal chains with the following group inclusion

$0 \subseteq 3 \subseteq 3^2 \subseteq 3^2 \times 2 \subseteq 3^2 \times 2^2 \subseteq 3^2 \times 2^3$

$0 \subseteq 3 \subseteq 3 \times 2 \subseteq 3 \times 2^2 \subseteq 3^2 \times 2^2 \subseteq 3^2 \times 2^3$

$0 \subseteq 3 \subseteq 3 \times 2 \subseteq 3 \times 2^2 \subseteq 3 \times 2^3 \subseteq 3^2 \times 2^3$

$0 \subseteq 3 \subseteq 3 \times 2 \subseteq 3^2 \times 2 \subseteq 3^2 \times 2^2 \subseteq 3^2 \times 2^3$

$0 \subseteq 2 \subseteq 3 \times 2 \subseteq 3 \times 2^2 \subseteq 3^2 \times 2^2 \subseteq 3^2 \times 2^3$

$0 \subseteq 2 \subseteq 3 \times 2 \subseteq 3 \times 2^2 \subseteq 3 \times 2^3 \subseteq 3^2 \times 2^3$

$0 \subseteq 2 \subseteq 2^2 \subseteq 2^3 \subseteq 2^3 \times 3 \subseteq 3^2 \times 2^3$

$0 \subseteq 2 \subseteq 2^2 \subseteq 3 \times 2^2 \subseteq 3^2 \times 2^2 \subseteq 3^2 \times 2^3$

$0 \subseteq 2 \subseteq 3 \times 2 \subseteq 3^2 \times 2 \subseteq 3^2 \times 2^2 \subseteq 3^2 \times 2^3$

$0 \subseteq 2 \subseteq 2^2 \subseteq 3 \times 2^2 \subseteq 3 \times 2^2 \subseteq 3^2 \times 2^3$

Also

$$\sum_{i=-1}^{3-1}(3-i) = \sum_{i=-1}^{2}(3-i) = (3+1)+(3)+(3-1)+(3-2) = 10$$

Theorem 2.6: Let $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$ where p and q are distinct primes. Then the number of maximal chains of G is $\sum_{i=-1}^{m-1} r_i(m-i)$ where $r_i = \frac{(n+i-1)!}{(n-2)!(1+i)!}$,$n \geq 2$.

Proof: For a fixed n we apply induction on m. Now let that $\mathbb{Z}_{p^n} + \mathbb{Z}_{q^k}$ has $\sum_{i=-1}^{k-1} r_i(k-i)$ maximal chains where $r_i = \frac{(n+i-1)!}{(n-2)!(1+i)!}$. We show that $\mathbb{Z}_{p^n} + \mathbb{Z}_{q^{k+1}}$ has $\sum_{i=-1}^{k} r_i(k+1-i)$ maximal chains where $r_i = \frac{(n+i-1)!}{(n-2)!(1+i)!}$.

As in Propositions 2.2 and 2.3, consider Fig. 6, we observe that total maximal chains as follows:

$p^n q^{k+1} \supset p^n q^k \supset \cdots$ $\qquad (n, k)$

$p^n q^{k+1} \supset p^{n-1} q^{k+1} \supset p^{n-1} q^k \cdots$ $\qquad (n-1, k)$

$p^n q^{k+1} \supset p^{n-1} q^{k+1} \supset p^{n-2} q^{k+1} \supset p^{n-2} q^k \cdots$ $\qquad (n-2, k)$

$p^n q^{k+1} \supset p^{n-1} q^{k+1} \supset p^{n-2} q^{k+1} \supset p^{n-3} q^{k+1} \supset p^{n-3} q^k \supset \cdots$ $\qquad (n-3, k)$

$\vdots$

$p^n q^{k+1} \supset p^{n-1} q^{k+1} \supset \cdots \supset p^{n-(n-1)} q^{k+1} \supset p^1 q^k \supset \cdots$ $\qquad (1, k)$

$p^n q^{k+1} \supset p^{n-1} q^{k+1} \supset \cdots \supset p^0 q^{k+1} \supset q^k \supset \cdots$ $\qquad (0, k)$

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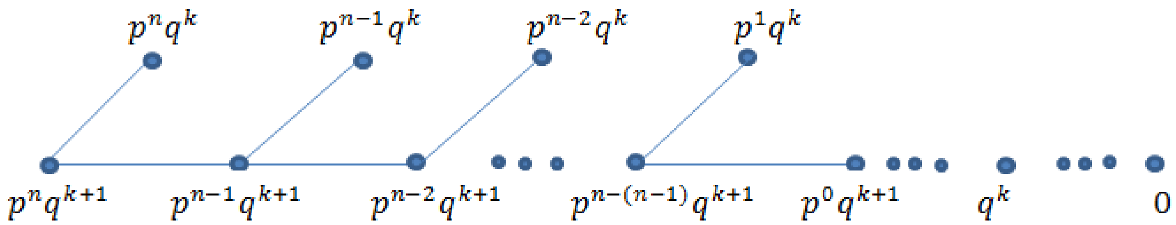

Fig. 6

For $2 \leq j \leq n$, $(n-j,k)$ yields $\sum_{i=-1}^{k} r_{(n-j)i}(k-i)$ maximal chains that $r_{(n-j)i} = \frac{(n-j+i-1)!}{(n-j-2)!(1+i)!}$. Also $(1,k)$ yields $k+1$ maximal chains and $(0,k)$ has only one maximal chain.

Thus, the total number of maximal chains is the double sum

$$N = \sum_{j=0}^{n-2} \sum_{i=-1}^{k-1} r_{(n-j)i}(k-i) + k + 1 + 1 = \sum_{j=0}^{n-2} \sum_{i=-1}^{k-1} \frac{(n-j+i-1)!}{(n-j-2)!\,(1+i)!}(k-i) + k + 1 + 1$$

$$= \sum_{j=0}^{n-2} \left[\frac{(n-j-2)!}{(n-j-2)!\,0!}(k+1) + \frac{(n-j-1)!}{(n-j-2)!\,1!}k + \frac{(n-j)!}{(n-j-2)!\,2!}(k-1) + \frac{(n-j+1)!}{(n-j-2)!\,3!}(k-2) + \cdots + \frac{(n-j+k-2)!}{(n-j-2)!\,k!}.1 \right] + k + 1 + 1 = \sum_{j=0}^{n-2} \left[k + 2 + (n-j-1)k + \frac{(n-j)(n-j-1)(k-1)}{2!} + \frac{(n-j+1)(n-j)(n-j-1)(k-2)}{3!} + \cdots + \frac{(n-j+k-2)\cdots(n-j-1)}{k!} \right] + k + 1 + 1$$

$$= (n-1)(k+1) + k \sum_{j=0}^{n-2} (n-j-1) + \frac{k-1}{2!} \sum_{j=0}^{n-2} (n-j)\,(n-j-1) + \frac{k-2}{3!} \sum_{j=0}^{n-2} (n-j+1)(n-j)\,(n-j-1) + \cdots + \frac{1}{k!} \sum_{j=0}^{n-2} (n-j+k-2)\cdots(n-j-1) + k + 1 + 1$$

Remark 2.7: For any positive integer n and k,

$$(n)(n+1)\cdots(n+k-1)+(n-1)(n)\cdots(n+k-2)+\cdots+2.3\cdots(k+1)+1.2\cdots k = \frac{1}{k+1}n(n+1)(n+2)\cdots(n+k)$$

For $k=2, (n)(n+1)+(n-1)n+\cdots+1.2=\frac{1}{3}n(n+1)(n+2)$

For $k=3, (n)(n+1)(n+2)+(n-1)n(n+1)+\cdots+1.2.3$

$$=\frac{1}{4}n(n+1)(n+2)(n+3)$$

So, with n replaced by $n-1$

$$\sum_{j=0}^{n-2}(n-j)\,(n-j-1)=(n-1)n+(n-2)(n-1)+\cdots+1.2=\frac{1}{3}(n-1)n(n+1)$$

$$\sum_{j=0}^{n-2}(n-j+1)(n-j)\,(n-j-1)=(n-1)n(n+1)+(n-2)(n-1)n+\cdots+1.2.3 = \frac{1}{4}(n-1)n(n+1)(n+2)$$

Thus, the total number of maximal chains is

$$N=(n-1)(k+1)+k\frac{n(n-1)}{2}+\frac{k-1}{2!}.\frac{(n-1)n(n+1)}{3} + \frac{k-2}{3!}\frac{(n-1)n(n+1)(n+2)}{4}+\cdots+\frac{1}{k!}.\frac{(n-1)n(n+1)\cdots(n+k-1)}{k+1} + k+1+1$$

$$\Rightarrow N=(n-1)(k+1)+\frac{k}{2!}n(n-1)+\frac{k-1}{3!}(n-1)n(n+1) + \frac{k-2}{4!}(n-1)n(n+1)(n+2)+\cdots + \frac{1}{(k+1)!}(n-1)n(n+1)\cdots(n+k-1)+k+1+1 \qquad (2.1)$$

On the other hand,

$$\sum_{i=-1}^{k} r_i(k+1-i) = \sum_{i=-1}^{k} \frac{(n+i-1)!}{(n-2)!\,(1+i)!}(k+1-i) = \frac{(n-2)!}{(n-2)!\,0!}(k+2)$$
$$+\frac{(n-1)!}{(n-2)!\,1!}(k+1) + \frac{n!}{(n-2)!\,2!}k + \cdots + \frac{(n+k-1)!}{(n-2)!\,(1+k)!}.1$$
$$= (k+2) + (n-1)(k+1) + \frac{n(n-1)}{2!}k + \cdots$$
$$+\frac{(n+k-1)(n+k-2)\cdots(n-1)}{(k+1)!} \qquad (2.2)$$

We see the right-hand side (2.1) and (2.2) are equal. So the number of maximal chains $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$ and $\sum_{i=-1}^{k} r_i(k+1-i)$ where $r_i = \frac{(n+i-1)!}{(n-2)!(1+i)!}$ are equal.

This completes the proof.

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